Harish-Chandra Research Institute

Algebraic Topology

Assignment 0

M.Sc. Mathematics, Year II • August–December 2026

Problem 1

Let X be a topological space, and let

$$\Delta_X = \{(x,x) : x \in X\} \subseteq X \times X$$

be the diagonal.

(a) Prove that X is Hausdorff if and only if Δ_X is closed in $X \times X$.

(b) Suppose that X is Hausdorff and that $f, g\colon Y \to X$ are continuous. Prove that their equalizer

$$\mathrm{Eq}(f,g) = \{y \in Y : f(y) = g(y)\}$$

is closed in Y.

Problem 2

Let $q\colon X \to Y$ be a surjective map, and equip Y with the quotient topology induced by q; thus, a subset $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open in X.

(a) Let Z be a topological space and let $g\colon Y \to Z$ be a map. Prove that

$$g \text{ is continuous} \quad \iff \quad g \circ q \text{ is continuous.}$$

(b) Let $f\colon X \to Z$ be continuous and suppose that f is constant on the fibres of q, meaning that

$$q(x) = q(x') \quad \implies \quad f(x) = f(x').$$

Prove that there is a unique continuous map $\overline{f}\colon Y \to Z$ such that

$$f = \overline{f} \circ q.$$

If q is not surjective, are the above two claims still true?

Problem 3

Let $\{X_i\}_{i \in I}$ be a family of topological spaces, let

$$P = \prod_{i \in I} X_i,$$

and let $\pi_i\colon P \to X_i$ denote the coordinate projections.

(a) Prove that the product topology on P is the coarsest topology for which every projection π_i is continuous.

(b) Let Y be a topological space and let $f\colon Y \to P$ be a map. Prove that

$$f \text{ is continuous} \quad \iff \quad \pi_i \circ f \text{ is continuous for every } i \in I.$$

(c) Given continuous maps $f_i\colon Y \to X_i$ for every $i \in I$, prove that there is a unique continuous map

$$f\colon Y \longrightarrow \prod_{i \in I} X_i$$

satisfying $\pi_i \circ f = f_i$ for every $i \in I$.

Problem 4

Let X be a topological space.

(a) Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of path-connected subspaces of X such that

$$\bigcap_{\lambda \in \Lambda} A_\lambda \neq \varnothing.$$

Prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is path connected.

(b) Deduce that if $A, B \subseteq X$ are path connected and $A \cap B \neq \varnothing$, then $A \cup B$ is path connected.

(c) Prove that a finite product of nonempty path-connected spaces is path connected.

Problem 5

(a) Let A be a connected topological space and let $f\colon A \to Y$ be continuous. Prove that $f(A)$ is connected.

(b) Prove that every path-connected topological space is connected.

(c) Fix $x \in X$ and define

$$C_x = \bigcup \{A \subseteq X : A \text{ is connected and } x \in A\}.$$

Prove the following statements:

(i) C_x is connected;

(ii) every connected subset of X containing x is contained in C_x;

(iii) if $C_x \cap C_y \neq \varnothing$, then $C_x = C_y$;

(iv) C_x is closed in X.