Harish-Chandra Research Institute

Algebraic Topology

Assignment 1

M.Sc. Mathematics, Year II • Date of Submission: 25/08/2026

Problem 1

1. Verify that topological spaces and continuous maps form a category **Top**.
2. Let $\mathbf{Top}_*$ denote the category whose objects are pointed topological spaces (X, x_0) and whose morphisms
$$f\colon (X, x_0) \longrightarrow (Y, y_0)$$
are continuous maps satisfying $f(x_0) = y_0$. Verify that $\mathbf{Top}_*$ is a category.

Problem 2

Let X and Y be topological spaces. Recall that two continuous maps $f, g\colon X \to Y$ are homotopic if there is a continuous map
$$H\colon X \times I \longrightarrow Y$$
such that
$$H(x, 0) = f(x) \qquad \text{and} \qquad H(x, 1) = g(x)$$
for every $x \in X$.

1. Prove that homotopy is an equivalence relation on $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$.

   If f is homotopic to g we denote it as $f \simeq g$.
2. Prove that if
$$f_0 \simeq f_1\colon X \to Y \qquad \text{and} \qquad g_0 \simeq g_1\colon Y \to Z,$$
then
$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$
3. Deduce that one obtains a category **hTop**, called the homotopy category, whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps.

Problem 3

Fix a topological space T.

1. Show that the assignment
$$X \longmapsto \mathrm{Hom}_{\mathbf{Top}}(T, X)$$
defines a covariant functor
$$\mathrm{Hom}_{\mathbf{Top}}(T, -)\colon \mathbf{Top} \longrightarrow \mathbf{Set}.$$

2. Show that the assignment
$$X \longmapsto \mathrm{Hom}_{\mathbf{Top}}(X, T)$$
defines a contravariant functor from **Top** to **Set**.

Problem 4

Let $I = [0, 1]$. Consider the functors
$$F, G\colon \mathbf{Top} \longrightarrow \mathbf{Top}$$
defined by
$$F(X) = X \times I, \qquad G(X) = X,$$
and
$$F(f) = f \times \mathrm{id}_I, \qquad G(f) = f.$$
For each $t \in I$, define
$$\iota_{t,X}\colon X \longrightarrow X \times I, \qquad \iota_{t,X}(x) = (x, t),$$
and define
$$p_X\colon X \times I \longrightarrow X, \qquad p_X(x, s) = x.$$

1. Prove that
$$\iota_t = \{\iota_{t,X}\}_X$$
is a natural transformation $G \Rightarrow F$.
2. Prove that
$$p = \{p_X\}_X$$
is a natural transformation $F \Rightarrow G$.
3. Compute the composition $p \circ \iota_t$.
4. Explain why $\iota_t \circ p$ is not equal to the identity natural transformation of F. Prove, however, that it is *naturally homotopic* to the identity.

Problem 5

A continuous map $f\colon X \to Y$ is called a homotopy equivalence if there exists a continuous map $g\colon Y \to X$ such that
$$g \circ f \simeq \mathrm{id}_X \qquad \text{and} \qquad f \circ g \simeq \mathrm{id}_Y\,.$$
In this case X and Y are said to be homotopy equivalent.

1. Prove that homotopy equivalence is an equivalence relation on topological spaces.
2. Prove that a topological space X is contractible if and only if it is homotopy equivalent to a one-point space.
3. Prove that every convex subset of $\mathbb{R}^n$ is contractible.

Problem 6

Let X be a topological space and let $x_0, x_1, x_2 \in X$. If $\alpha\colon I \to X$ is a path from x_0 to x_1, define the reverse path by
$$\overline{\alpha}(t) = \alpha(1 - t).$$

If α is a path from x_0 to x_1 and β is a path from x_1 to x_2, define their concatenation by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

1. Prove that $\alpha * \beta$ is continuous.
2. Show that concatenation is not strictly associative as an operation on parametrized paths by providing an explicit counterexample where $(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$.
3. Construct an explicit homotopy relative to the endpoints between

$$(\alpha * \beta) * \gamma \qquad \text{and} \qquad \alpha * (\beta * \gamma).$$

Problem 7

Let (X, x_0) be a pointed topological space. Let $\pi_1(X, x_0)$ denote the set of homotopy classes relative to the endpoints of loops based at x_0.

Prove carefully that the operation

$$[\alpha] \cdot [\beta] = [\alpha * \beta]$$

is well defined and makes $\pi_1(X, x_0)$ into a group.

Problem 8

Let

$$f \colon (X, x_0) \longrightarrow (Y, y_0)$$

be a pointed continuous map. Define

$$f_* \colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \qquad f_*([\alpha]) = [f \circ \alpha].$$

1. Prove that f_* is well defined.
2. Prove that f_* is a group homomorphism.
3. Prove that

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}.$$

4. For pointed maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0),$$

prove that

$$(g \circ f)_* = g_* \circ f_*.$$

5. Conclude that the fundamental group defines a functor

$$\pi_1 \colon \mathbf{Top}_* \longrightarrow \mathbf{Grp}.$$

Problem 9

Let $x_0, x_1 \in X$, and let λ be a path from x_0 to x_1. Define

$$c_\lambda \colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by

$$c_\lambda([\alpha]) = [\,\overline{\lambda} * \alpha * \lambda\,].$$

1. Prove that c_λ is a well-defined group isomorphism. What is $(c_\lambda)^{-1}$?
2. Suppose that μ is another path from x_0 to x_1. Show that the composition $c_\lambda \circ (c_\mu)^{-1}$ is given by an inner automorphism of $\pi_1(X, x_1)$. Explicitly identify the element of $\pi_1(X, x_1)$ defining this conjugation.

Problem 10

1. Let
$$f, g\colon (X, x_0) \longrightarrow (Y, y_0)$$
be pointed maps that are homotopic through a pointed homotopy. Thus, suppose that there exists a homotopy
$$H\colon X \times I \longrightarrow Y$$
such that
$$H(x, 0) = f(x), \qquad H(x, 1) = g(x), \qquad H(x_0, t) = y_0$$
for all $x \in X$ and $t \in I$. Prove that
$$f_* = g_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$
2. Deduce that a pointed homotopy equivalence induces an isomorphism of fundamental groups.
3. Prove that
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$
4. Compute
$$\pi_1(X \times \mathbb{R}^n, (x_0, 0))$$
in terms of $\pi_1(X, x_0)$.