Harish-Chandra Research Institute

Algebraic Topology

Assignment 2

M.Sc. Mathematics, Year II •

Problem 1

Is every group homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ induced by a continuous map $f : S^1 \to S^1$? Justify your answer rigorously.

Problem 2

Let $F : X \times I \to X$ be a homotopy from f_0 to f_1, where we follow the notation that $F(x,t) := f_t(x)$. Further assume that $f_0(x) = f_1(x) = x$ for all $x \in X$. Show that for any $x_0 \in X$, the loop $\gamma(t) = f_t(x_0)$ represents an element in the center of $\pi_1(X, x_0)$.

Problem 3

Let $X := \mathbb{R}^n \setminus \{x_1, \dots, x_m\}$, where $x_i \in \mathbb{R}^n$ are distinct points. Compute $\pi_1(X)$ for all integers $n \geq 2$.

Problem 4

Let $p : \tilde{X} \to X$ be a covering space. Let $A \subset X$ be a subspace equipped with the subspace topology, and let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering space. Furthermore, assume that the fiber $p^{-1}(x)$ is a finite set for all $x \in X$. Prove that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

Problem 5

Let X be a path-connected, locally path-connected topological space. Suppose that $\pi_1(X)$ is a finite group. Prove that every continuous map $f : X \to S^1$ is nullhomotopic.

Problem 6

Classify all connected, two-sheeted covering spaces of the figure-eight space $S^1 \vee S^1$ up to isomorphism.

Problem 7

Let G be a finitely generated abelian group. Construct a path-connected topological space X such that $\pi_1(X) \cong G$.

Problem 8

Let G be a path-connected topological group with identity element e. Prove that the fundamental group $\pi_1(G, e)$ is abelian.

Problem 9

Let X be the complement in $\mathbb{R}^3$ of k distinct lines passing through the origin. Compute $\pi_1(X)$.

Problem 10

Let Y be a simply connected, locally path-connected space. Suppose a discrete group G acts properly discontinuously on Y. Prove that the fundamental group of the orbit space, $\pi_1(Y/G)$, is isomorphic to G.

Problem 11

Let D^2 be the closed unit disk and S^1 the unit circle. Prove that there is no continuous retraction from the solid torus $D^2 \times S^1$ to its boundary torus $S^1 \times S^1$.