Harish-Chandra Research Institute

Algebraic Topology

Assignment 3

M.Sc. Mathematics, Year II •

Problem 1

Decide whether the following statements are TRUE or FALSE, providing justification for your answers.

(a) If $p : E \longrightarrow X$ is a covering and the induced homomorphism $p_* : \pi_1(E, e) \longrightarrow \pi_1(X, p(e))$ is surjective, then p is a homeomorphism.

(b) If a space X has a universal cover, then X is semi-locally simply connected.

(c) The diagonal circle $C := \{(x, y) \in S^1 \times S^1 \mid x = y\}$ is a retract of the torus $S^1 \times S^1$.

Problem 2

Let $L_1, \ldots, L_m$ be m lines passing through the origin in $\mathbb{R}^3$ and $X = \bigcup_{j=1}^{m} L_j$. Show that the sphere S^2 with $2m$ points removed is a deformation retract of $\mathbb{R}^3 - X$. Hence, show that $\pi_1(\mathbb{R}^3 - X) \cong F_{2m-1}$.

Problem 3

Describe the real projective plane $\mathbb{R}P^2$ as a CW-complex. Furthermore, show that the projection map $q : S^n \longrightarrow \mathbb{R}P^n$ is a covering map. Also, show that the group of covering transformations of this covering map is $\mathbb{Z}_2$.

Problem 4

Show that the following statements are equivalent for a continuous map $f : S^n \longrightarrow X$:

(a) f is homotopic to a constant map.

(b) f can be extended to a map $D^{n+1} \longrightarrow X$.

Recall that D^{n+1} denotes the set of elements $x \in \mathbb{R}^{n+1}$ with $\|x\| \leq 1$.

Problem 5

1. Construct a topological space whose fundamental group is $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ (for any two positive integers m, n).
2. Construct a covering space $P : Y \longrightarrow X$ whose deck transformation group $\mathrm{Aut}(Y/X)$ is isomorphic to a given finitely generated abelian group G.

Problem 6

Show that a covering map $p : E \longrightarrow X$ is regular if and only if for each closed path $\alpha : [0,1] \longrightarrow X$, either every lifting $\tilde{\alpha}$ of α is a closed path or no lifting $\tilde{\alpha}$ of α is a closed path.

Problem 7

Let $F_2 = \mathbb{Z} * \mathbb{Z}$ be the free product of two copies of $\mathbb{Z}$, the first copy generated by a and the second generated by b. Let $H = \langle\langle \{ab, ba, a^4\} \rangle\rangle$ be the normal closure of $\{ab, ba, a^4\}$ in F_2. Show that H has finite index in F_2.

Problem 8

Let $T = S^1 \times S^1$. There is an isomorphism of $\pi_1(T, b_0 \times b_0)$ with $\mathbb{Z} \times \mathbb{Z}$ induced by projections of T onto its two factors. Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times 0$ and $0 \times n$, where m and n are positive integers.

Problem 9

Let X be a simply connected space and let $p : \tilde{X} \longrightarrow X$ be a covering space of X with $\tilde{X}$ connected. Prove that p must be a homeomorphism.