Harish-Chandra Research Institute

Algebraic Topology

Assignment 4

M.Sc. Mathematics, Year II •

Problem 1

Construct an explicit deformation retraction from $\mathbb{R}^n \setminus \{0\}$ onto S^1.

Problem 2

Show that a retract of a contractible space is contractible.

Problem 3

Show that a space X is contractible if and only if every map $f : X \to Y$ for arbitrary Y is nullhomotopic.

Problem 4

Let X be a path-connected topological space. Show that $\pi_1(X)$ is abelian if and only if the base change homomorphism β_h depends only on the endpoint of the path h.

Problem 5

If X_0 is a path component of a topological space X containing the base point x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism on π_1 (basepoint being x_0).

Problem 6

Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.

Problem 7

Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Show that for any $x_0 \in X$, $t \mapsto f_t(x_0)$ considered as a loop in $\pi_1(X, x_0)$ lies in the center.

Problem 8

Let X be the quotient space of S^2 obtained by identifying the north and south poles to a point. Put a cell complex structure on X and use this to compute its π_1.

Problem 9

Show that the join $X * Y$ of two non-empty spaces X and Y is simply connected if X is path-connected.

Problem 10

Let $\widetilde{X}$ and $\widetilde{Y}$ be two simply connected covering spaces of path-connected, locally path-connected spaces X and Y. Show that if $X \cong Y$ then $\widetilde{X} \cong \widetilde{Y}$.

Problem 11

Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion is injective.

Problem 12

Show that $H_0(X, A) = 0$ iff A meets every path component of X.

Problem 13

Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path component of X contains at most one path component of A.

Problem 14

Show that for the subspace $\mathbb{Q} \in \mathbb{R}$ the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is a free abelian group. Find a basis.

Problem 15

Describe a CW structure on the following spaces and compute their homology:

1. $\mathbb{C}P^n$.
2. M_g: a closed orientable surface of genus g.
3. $\mathbb{R}P^n$.

Problem 16

If a finite CW complex X is a union of subcomplexes A and B then show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Problem 17

Given a map $f : S^{2n} \to S^{2n}$ show that there exists some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

Problem 18

Let M_g denote the surface of genus g with its standard embedding in $\mathbb{R}^3$. Let R denote the compact region in $\mathbb{R}^3$ that it bounds.

Let X denote two copies of R glued along the boundary surface M_g. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R.