Harish-Chandra Research Institute

Introduction to Algebraic Geometry

Assignment 1

M.Sc. Mathematics, Year II •

Rings are commutative with identity, and k denotes a field. For an algebraic set $V \subseteq \mathbb{A}^n(k)$, $I(V)$ is its ideal and $\Gamma(V) = k[X_1, \dots, X_n]/I(V)$ its coordinate ring.

Problem 1

Let k be any field. Prove that $k[X]$ contains infinitely many irreducible monic polynomials.

Problem 2

Prove that the algebraic subsets of $\mathbb{A}^1(k)$ are exactly the finite subsets together with $\mathbb{A}^1(k)$ itself.

Problem 3

Prove that each of the following is an algebraic set:

(a) $\{(t, t^2, t^3) \in \mathbb{A}^3(k) \ : \ t \in k\}$;
(b) $\{(\cos t, \sin t) \in \mathbb{A}^2(\mathbb{R}) \ : \ t \in \mathbb{R}\}$;
(c) the set of points of $\mathbb{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy $r = \sin\theta$.

Problem 4

Prove that none of the following is an algebraic set:

(a) $\{(x, y) \in \mathbb{A}^2(\mathbb{R}) \ : \ y = \sin x\}$;
(b) $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) \ : \ |z|^2 + |w|^2 = 1\}$, where $|x + iy|^2 = x^2 + y^2$ for $x, y \in \mathbb{R}$;
(c) $\{(\cos t, \sin t, t) \in \mathbb{A}^3(\mathbb{R}) \ : \ t \in \mathbb{R}\}$.

Problem 5

Let $V \subseteq \mathbb{A}^n(k)$ and $W \subseteq \mathbb{A}^m(k)$ be algebraic sets. Prove that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \ : \ (a_1, \dots, a_n) \in V, \ (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbb{A}^{n+m}(k)$; it is called the *product* of V and W.

Problem 6

Let V, W be algebraic sets in $\mathbb{A}^n(k)$. Prove that $V = W$ if and only if $I(V) = I(W)$.

Problem 7

Prove that $F = Y^2 + X^2(X-1)^2 \in \mathbb{R}[X, Y]$ is irreducible, while $V(F)$ is reducible.

Problem 8

(a) Find the irreducible components of $V(Y^2 - XY - X^2Y + X^3)$ in $\mathbb{A}^2(\mathbb{R})$, and also in $\mathbb{A}^2(\mathbb{C})$.
(b) Do the same for $V(Y^2 - X(X^2 - 1))$ and for $V(X^3 + X - X^2Y - Y)$.

Problem 9

Let K be a field and $L = K(X)$ the field of rational functions in one variable over K. Prove that every element of L that is integral over $K[X]$ already lies in $K[X]$.

Problem 10

Let $V \subseteq \mathbb{A}^n$ be a nonempty variety. Prove that the following are equivalent:

(i) V is a point;
(ii) $\Gamma(V) = k$;
(iii) $\dim_k \Gamma(V) < \infty$.

Problem 11

(a) Prove that $\{(t, t^2, t^3) \in \mathbb{A}^3(k) \ : \ t \in k\}$ is an affine variety.
(b) Prove that $V(XZ - Y^2,\ YZ - X^3,\ Z^2 - X^2Y) \subseteq \mathbb{A}^3(\mathbb{C})$ is a variety. *(Hint: $Y^3 - X^4$, $Z^3 - X^5$, $Z^4 - Y^5 \in I(V)$; find a polynomial map from $\mathbb{A}^1(\mathbb{C})$ onto V.)*

Problem 12

A set $V \subseteq \mathbb{A}^n(k)$ is a *linear subvariety* of $\mathbb{A}^n(k)$ if $V = V(F_1, \ldots, F_r)$ for some polynomials F_i of degree 1.

(a) Prove that if T is an affine change of coordinates on $\mathbb{A}^n$, then V^T is again a linear subvariety of $\mathbb{A}^n(k)$.
(b) Assume $V \neq \emptyset$. Prove that there is an affine change of coordinates T of $\mathbb{A}^n$ with $V^T = V(X_{m+1}, \ldots, X_n)$; in particular V is a variety. *(Hint: induct on r.)*
(c) Prove that the integer m occurring in part (b) does not depend on the choice of T. It is called the *dimension* of V, and V is then isomorphic as a variety to $\mathbb{A}^m(k)$.