1. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a quotient map. Show that if each set $\pi^{-1}(\{y\})$ is connected and if $\mathcal{Y}$ is connected then $\mathcal{X}$ is connected.

2. Let $f : S^1 \to \mathbb{R}$ be a continous map. Show that there exists a point x of S^1 such that $f(x) = f(-x)$.

3. Let G be a topological group and let X_1 and X_2 be two closed subsets of G whre X_2 is compact. Show that $X_1 \cdot X_2$ is a closed subset of G.

4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a continous map of compact metric spaces. Show that f is uniformly continuous.

5. A topological space $\mathcal{A}$ is said to have no isolated points if there does not exist any singleton set $\{a\} \subset \mathcal{A}$ which is open in $\mathcal{A}$. Let $\mathcal{X}$ be a compact Hausdorff topological space that does not have any isolated points. What are the possibilities for the cardinality of $\mathcal{X}$?

6. A topological space $\mathcal{X}$ is said to be limit point compact if every infinite subset of $\mathcal{X}$ has a limit point. Prove that if $\mathcal{X}$ is compact it is limit point compact.

7. Let G be a locally compact topological group and let H be a subgroup of G. Show that G/H is locally compact.

8. Let $\mathcal{X}$ be a topological space that has a countable basis. Show that $\mathcal{X}$ has a countable dense subset. A topological space that has a countable dense subset is called **separable**.

9. (a) Let $\mathcal{X}$ be a topological space that has a countable basis. Show that every open covering of $\mathcal{X}$ has a countable subcover. A topological space in which every cover has a countable subcover is called a **Lindelöf** space.

   (b) Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces and let $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$. Which of the following statements are true and why

      i. If $\mathcal{X}$ is Lindelöf and $\mathcal{Y}$ is Lindelöf then $\mathcal{Z}$ is Lindelöf.
      ii. If $\mathcal{X}$ is Lindelöf and $\mathcal{Y}$ is compact then $\mathcal{Z}$ is Lindelöf.
      iii. If $\mathcal{X}$ is compact and $\mathcal{Y}$ is compact then $\mathcal{Z}$ is compact.

10. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a surjective continuous map of topological spaces which is closed. Show that if $\mathcal{X}$ is normal then $\mathcal{Y}$ is normal.

11. Let $\mathcal{X}$ be a topological space and let $\mathcal{Y}$ be a compactification of $\mathcal{X}$. Let $\beta(\mathcal{X})$ be the Stone-Čech compactification of $\mathcal{X}$. Show that there exists a continuous closed map $g : \beta(\mathcal{X}) \to \mathcal{Y}$ which is the identity on $\mathcal{X}$.

12. Let $\mathcal{X}$ be a topological space and G a topological group. A G-action on $\mathcal{X}$ is defined by the following :

   (a) A continous map $f : G \times \mathcal{X} \to \mathcal{X}$ where we shall denote $f(g, x) = g \cdot x$.
   (b) $e \cdot x = x$ where e denotes the identity of G.
   (c) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$. Let $R \subset X \times X$ be defined as follows:

$$R := \{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} \mid x_1 = g \cdot x_2 \text{ for some } g \in G\}. \tag{1}$$

   Check that R defines an equivalence relation on $\mathcal{X}$. We shall denote $\mathcal{X}/R$ by $\mathcal{X}/G$ where it is a topoloogical space with the quotient topology for the quotient map $\pi : \mathcal{X} \to \mathcal{X}/G$. Check that if $\mathcal{X}$ is T_2 or T_3 or T_4 then so is $\mathcal{X}/G$.

11/08/2023