Harish-Chandra Research Institute

Advanced Algebraic Geometry (Scheme Theory)

Assignment 1

M.Sc. Mathematics, Year II •

Rings are commutative with identity. For a scheme X and $x \in X$, write $\mathcal{O}_{X,x}$ for the local ring at x, $\mathfrak{m}_x$ for its maximal ideal, and $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Problem 1

Let A be a ring, $X = \operatorname{Spec} A$, $f \in A$, and $D(f) = X \setminus V\big((f)\big)$. Prove that the locally ringed space $\big(D(f),\ \mathcal{O}_X|_{D(f)}\big)$ is isomorphic to $\operatorname{Spec} A_f$.

Problem 2

A scheme $(X, \mathcal{O}_X)$ is *reduced* if $\mathcal{O}_X(U)$ has no nonzero nilpotents for every open $U \subseteq X$. For a ring A, let A_{red} denote the quotient of A by its ideal of nilpotent elements.

(a) Prove that X is reduced if and only if $\mathcal{O}_{X,P}$ has no nonzero nilpotents for every $P \in X$.
(b) Let $\mathcal{O}_X^{\mathrm{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\mathrm{red}}$. Prove that $X_{\mathrm{red}} := (X, \mathcal{O}_X^{\mathrm{red}})$ is a scheme, and construct a morphism $X_{\mathrm{red}} \to X$ that is a homeomorphism on underlying topological spaces.
(c) Let $f\colon X \to Y$ be a morphism of schemes with X reduced. Prove that f factors uniquely as $X \xrightarrow{g} Y_{\mathrm{red}} \to Y$, the second map being the canonical one.

Problem 3

For a scheme X and $x \in X$, the *Zariski tangent space* is $T_x = \big(\mathfrak{m}_x/\mathfrak{m}_x^2\big)^\vee$, the dual taken over $k(x)$. Let X be a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers. Prove that the k-morphisms $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to X$ correspond bijectively to the pairs (x, t) with $x \in X$ satisfying $k(x) = k$ and $t \in T_x$.

Problem 4

Let X be a topological space and $Z \subseteq X$ an irreducible closed subset. A point ζ is a *generic point* of Z if $Z = \overline{\{\zeta\}}$. Prove that every nonempty irreducible closed subset of a scheme has a unique generic point.

Problem 5

(a) Let $f\colon X \to Y$ be a morphism of schemes and let $\{U_i\}$ be an open cover of Y such that $f^{-1}(U_i) \to U_i$ is an isomorphism for every i. Prove that f is an isomorphism.
(b) For $f \in \Gamma(X, \mathcal{O}_X)$, set $X_f = \{x \in X :\ f_x \notin \mathfrak{m}_x\}$. Prove that X is affine if and only if there exist $f_1, \dots, f_r \in \Gamma(X, \mathcal{O}_X)$ generating the unit ideal such that each X_{f_i} is affine.